

# Killing and Noether Symmetries of Plane Symmetric Spacetime

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## Abstract

This paper is devoted to investigate the Killing and Noether symmetries of static plane symmetric spacetime. For this purpose, five different cases have been discussed. The Killing and Noether symmetries of Minkowski spacetime in cartesian coordinates are calculated as a special case and it is found that Lie algebra of the Lagrangian is 10 and 17 dimensional respectively. The symmetries of Taub's universe, anti-deSitter universe, self similar solutions of infinite kind for parallel perfect fluid case and self similar solutions of infinite kind for parallel dust case are also explored. In all the cases, the Noether generators are calculated in the presence of gauge term. All these examples justify the conjecture that Killing symmetries form a subalgebra of Noether symmetries [1].

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# 1 Introduction

Symmetry is an important phenomenon to understand the universe. Most of the stars are assumed to have sphere like symmetry. Cylindrical and plane symmetries may be used to investigate the gravitational waves. Since general relativity (GR) and other modified theories such as  $f(R)$  [2],  $f(T)$  [3],  $f(R, T)$  [4] theories of gravity, Horva-Lifhitz [5] gravity and Brans-Dicke theory [6] etc. are highly non linear, so the solutions of their field equations can be understood through their symmetries. These symmetries are given by isometries or Killing vectors (KVs) of the spacetime. An isometry is a direction along which Lie derivative of the metric tensor is zero. It would be worthwhile to mention here that many GR solutions do have some symmetry [7]. The problem of localization of energy momentum in GR can be addressed using symmetries. There always exists a timelike Killing Vector (KV) for static spacetimes which may be used to define the energy of a particle using the equation  $E = \alpha.p$ , where  $\alpha$  is the timelike KV and  $p$  represents the momentum of the particle [8]. Similarly the rotational symmetries of static and spherically symmetric spacetime such as an exterior of a star provide conservation of angular momentum. Thus symmetries play an important role in many physical applications of gravitational fields.

Another systematic approach to find the conserved quantities [9, 10] of the variational problems is given by Emmy Noether. She gave a relationship between the conservation laws and symmetries [10], known as Noether symmetries. Conservation laws are very important in the field of differential equations. They help in computing the unknown exponent in the similarity solution which may not be obtained from the homogeneous boundary conditions [11]. The conserved quantities are also proved to be helpful to control numerical errors in the integration process of partial differential equations (PDEs). The celebrated Noether's theorem [10] states that for every infinitesimal generator of Lagrangian symmetry, there exist a conserved quantity. The invariance of time translation corresponds to the conservation of energy. Thus it is hoped that the Noether symmetry approach may become helpful to define the energy content of gravitational waves. This theorem also suggests that constants of motion for a given Lagrangian are related to its symmetry transformations [12].

The importance of Noether symmetries can be seen from the fact that these symmetries may recover some lost conservation laws and symmetry generators of spacetimes [13]. Sharif and Saira [14] calculated the energy

contents of colliding plane waves by using approximate symmetries. It has been concluded that there does not exist any nontrivial first order symmetry for plane electromagnetic and gravitational waves. The same authors [15] used approximate Lie symmetries to explore the energy contents of Bardeen model and stringy charged black hole type solutions. In a recent paper [16], we have investigated  $f(R)$  theory of gravity using Noether symmetries in the presence of gauge term. For this purpose, we discussed Noether symmetry generators for spherically symmetric spacetimes and Friedmann Robertson Walker universe along with the importance and stability criteria of some well known  $f(R)$  gravity models. Lie symmetry methods for differential equations were used to explore the symmetries of a perturbed Lagrangian for a plane symmetric gravitational wave like spacetime [17]. In recent years, many authors have studied Noether and Killing symmetries in different contexts.

Hussain et al. [18] investigated second order approximate symmetries of the geodesic equations for the Reissner-Nordström metric and it was concluded that energy must be re-scaled. The same authors [19] gave a proposal to determine the energy content of gravitational waves using approximate symmetries of differential equations. They also investigated exact and approximate Noether symmetries of the geodesic equations for the charged Kerr spacetime in [20]. The re-scaling of the energy contents was done and it was conjectured that for any spacetime, the conformal KVs form a subalgebra of the symmetries of the Lagrangian that minimizes arc length. Later on, Hussain [21] gave a counter example to prove that the conjecture was not true in general for any spacetime with non-zero curvature. Bokhari et al. [1] studied the symmetry generators of Killing vectors and Noether symmetries. It was concluded that the Noether symmetries contained the set of Killing symmetries. Thus they gave a conjecture that the Noether symmetries form a bigger set of Lie algebra and the Killing symmetries are a subset of that algebra. A similar comparison of Killing and Noether symmetries was given for conformally flat Friedmann metric [22]. Hall [23] gave a general relation connecting the dimensions of the Killing algebra, the orbits and the isotropies of the spacetime.

A plane symmetric spacetime may be considered as a Lorentzian manifold which possess a physical energy momentum tensor. It admits the minimal isometry group  $SO(2) \times \mathbb{R}^2$  such that the group orbits form space-like surfaces with constant curvature. Feroze et al. [24] discussed a complete classification of plane symmetric Lorentzian manifolds according to their additional isometries by solving the Killing equations. Classification of static

plane symmetric spacetimes according to their matter collineations is given by Sharif [25]. In this paper, we are focussed to investigate the Killing and Noether symmetries of plane symmetric spacetime. For this purpose we take five different cases namely Minkowski spacetime (as a special case), Taub's universe, anti-deSitter universe, self similar solutions of infinite kind for parallel perfect fluid and self similar solutions of infinite kind for parallel dust case. The plan of the paper is as follows: In second section, we give some basics of KVs and Noether symmetries. Sections 3 and 4 are used to calculate the Killing and Noether symmetries of the spacetimes mentioned above. In the last section we summarize and compare the results.

## 2 Killing and Noether Symmetries

A transformation that leaves the object (in our case, the metric) invariant is called symmetry. However, the precise definition of the symmetry is as follows: A symmetry of an object is a diffeomorphic mapping of the object to itself that preserves the structure of the object and leaves it invariant. Now let the mapping  $T$  of an object  $P$  be

$$T(\lambda) : P \rightarrow P,$$

where  $\lambda$  parameterizes the symmetry. For example, the Minkowski spacetime is time invariant and its translational invariance is given by

$$T(\lambda) : (t, x^i) \rightarrow (t + \lambda, x^i).$$

Mainly there exist two types of transformations, discrete and continuous. Reflection is an example of discrete transformation while translations and rotations are continuous transformations. In GR, continuous transformations are most important as these can be obtained in a systematical and mathematical way by finding the Killing vectors of a spacetime. The transformation which leaves the metric tensor  $g_{ij}$  invariant, is called an isometry. Thus an isometry is a direction along which the metric tensor is Lie transported, i.e. if the vector field  $K$  is an isometry then

$$\mathcal{L}_K g_{ij} = 0,$$

where for a four dimensional spacetime,  $i, j = 0, 1, 2, 3$ . In index notation, we have

$$K_{i;j} + K_{j;i} = 0,$$

where semi colin denotes covariant derivative and this equation is known as Killing's equation. Any vector satisfying this equation is called Killing vector. Thus  $K$  is a Killing vector field if and only if

$$K_{(i;j)} = 0.$$

It is mentioned here that the symmetries of a manifold may be characterized by its KVs and a finite dimensional Lie group is formed [26].

Noether symmetries are also known as symmetries of a Lagrangian. The method for calculating the Noether gauge symmetries by using the Lagrangian is given below.

Let

$$ds^2 = g_{ij}dx^i dx^j \quad (1)$$

be a line element then the vector field  $X$  for (1) will be

$$X = \xi(s, x^i) \frac{\partial}{\partial s} + \eta^j(s, x^i) \frac{\partial}{\partial x^j}. \quad (2)$$

The Lagrangian  $L$  for (1) can be computed by using

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \quad (3)$$

where dot denotes derivative with respect to  $s$ . The Noether gauge symmetry is given by the equation

$$X^{[1]}(L) + LD_s(\xi) = D_s(G), \quad (4)$$

where  $G$  is a gauge function and  $X^{[1]}$  is the first prolongation given by

$$X^{[1]} = X + (\eta^j_{,s} + \eta^j_{,i} \dot{x}^i - \xi_{,s} \dot{x}^j - \xi_{,i} \dot{x}^i \dot{x}^j) \frac{\partial}{\partial \dot{x}^j} \quad (5)$$

and  $D_s$  is defined as

$$D_s = \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i}.$$

### 3 Killing Symmetries of Plane Symmetric space-time

The general static plane symmetric spacetime is given by [8]

$$ds^2 = A(x)dt^2 - C(x)dx^2 - B(x)(dy^2 + dz^2). \quad (6)$$

For simplicity we take the coefficient of  $dx^2$  equal to unity so that the above spacetime becomes

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2). \quad (7)$$

For (7), the Killing equations turn out to be

$$k^1_{,x} = 0, \quad (8)$$

$$k^2_{,z} + k^3_{,y} = 0, \quad (9)$$

$$A_{,x}k^1 + 2Ak^0_{,t} = 0, \quad (10)$$

$$Ak^0_{,x} - k^1_{,t} = 0, \quad (11)$$

$$Ak^0_{,y} - Bk^2_{,t} = 0, \quad (12)$$

$$Ak^0_{,z} - Bk^3_{,t} = 0, \quad (13)$$

$$k^1_{,y} + Bk^2_{,x} = 0, \quad (14)$$

$$k^1_{,z} + Bk^3_{,x} = 0, \quad (15)$$

$$B_{,x}k^1 + 2Bk^2_{,y} = 0, \quad (16)$$

$$B_{,x}k^1 + 2Bk^3_{,z} = 0. \quad (17)$$

We suppose the vector field  $X$  of the form

$$X = k^0(t, x, y, z) \frac{\partial}{\partial t} + k^1(t, x, y, z) \frac{\partial}{\partial x} + k^2(t, x, y, z) \frac{\partial}{\partial y} + k^3(t, x, y, z) \frac{\partial}{\partial z}. \quad (18)$$

Now we solve these equations simultaneously for different values of metric coefficients  $A(x)$  and  $B(x)$ . We take Minkowski spacetime as a special case when  $A(x) = B(x)$ . The second case is the Taub's universe [27] in which  $A(x) = x^{-2/3}$  and  $B(x) = x^{4/3}$ . The third case gives the anti-deSitter universe [24] for  $A(x) = e^{2x} = B(x)$ . The last two cases are self similar solutions of infinite Kind for parallel perfect fluid case and dust case [28] when  $A(x) = 1$ ,  $B(x) = e^{2x}$  and  $A(x) = x^2$ ,  $B(x) = 1$  respectively.

### 3.1 Minkowski Spacetime

Here we shall investigate the Killing symmetries by solving Eqs. (8)-(17) for  $A(x) = B(x) = 1$ . After some tedious calculations, we obtain the solution

$$\begin{aligned} k^0 &= c_1x + c_7y + c_8z + c_{10}, \\ k^1 &= c_1t + c_4y + c_2z + c_3, \\ k^2 &= c_7t - c_4x + c_5z + c_6, \\ k^3 &= c_8t - c_2x - c_5y + c_9. \end{aligned}$$

Thus the Killing symmetries turn out to be

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad (19)$$

$$X_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad X_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad X_6 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad (20)$$

$$X_7 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}, \quad X_8 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \quad X_9 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}, \quad (21)$$

where these symmetries represent translations in  $t$ ,  $x$ ,  $y$ ,  $z$ , rotations in  $xy$ ,  $yz$ ,  $zx$  axis and Lorentz rotations in  $x$ ,  $y$ ,  $z$  respectively. These symmetry generators provides conservation laws for energy, spin angular momentum and linear momentum [29]. Minkowski spacetime forms maximal algebra having 10 Killing symmetries which is known as Poincare algebra  $so(1, 3) \oplus \mathbb{R}^4$ , where  $\oplus$  represents a semi direct sum [8].

### 3.2 Taub's Universe

Now we explore the Taub's universe [30]. In this case the simultaneous solutions of Eqs.(8)-(17) for  $A(x) = x^{-2/3}$ ,  $B(x) = x^{4/3}$  becomes

$$\begin{aligned} k^0 &= c_1, & k^1 &= 0, \\ k^2 &= c_2z + c_3, & k^3 &= c_4 - c_2y. \end{aligned}$$

Here we obtain four Killing symmetries  $X_0$ ,  $X_2$ ,  $X_3$  and  $X_5$ . It is mentioned here that these four independent KVs provide the minimal algebra associated with the plane symmetric spacetimes [8].

### 3.3 Anti-deSitter Universe

For anti-deSitter universe, the solution of the killing equations turns out to be

$$\begin{aligned}
k^0 &= \frac{1}{2} \left( -e^{-2x} - t^2 - y^2 - z^2 \right) c_1 - c_2 t z - c_3 t - c_4 t y + c_5 y + c_8 z + c_{10}, \\
k^2 &= \frac{1}{2} \left( e^{-2x} - t^2 - y^2 + z^2 \right) c_4 - c_1 t y - c_3 y - c_2 z y + c_5 t + c_7 z + c_6, \\
k^3 &= \frac{1}{2} \left( e^{-2x} - t^2 + y^2 - z^2 \right) c_2 - c_1 t z - c_3 z - c_4 z y + c_8 t - c_7 y + c_9, \\
k^1 &= c_1 t + c_4 y + c_2 z + c_3.
\end{aligned}$$

Here we get 10 Killing symmetries as follows:

$$X_0, \quad X_2, \quad X_3, \quad X_5, \quad X_8, \quad X_9, \quad (22)$$

$$X_{10} = \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z}, \quad (23)$$

$$X_{11} = \frac{1}{2} \left( -e^{-2x} - t^2 - y^2 - z^2 \right) \frac{\partial}{\partial t} + t \left( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right), \quad (24)$$

$$X_{12} = \frac{1}{2} \left( e^{-2x} - t^2 - y^2 + z^2 \right) \frac{\partial}{\partial y} + y \left( \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} - z \frac{\partial}{\partial z} \right), \quad (25)$$

$$X_{13} = \frac{1}{2} \left( e^{-2x} - t^2 + y^2 - z^2 \right) \frac{\partial}{\partial z} + z \left( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - t \frac{\partial}{\partial t} \right). \quad (26)$$

### 3.4 Self Similar Solution of Infinite Kind for Parallel Perfect Fluid Case

Here the solution becomes

$$\begin{aligned}
k^0 &= c_1, \\
k^1 &= c_3 y + c_7 z + c_2, \\
k^2 &= \frac{1}{2} \left( e^{-2x} - y^2 + z^2 \right) c_3 - c_2 y - c_4 z - c_7 z y + c_6, \\
k^3 &= \frac{1}{2} \left( e^{-2x} + y^2 - z^2 \right) c_7 - c_2 z - c_3 z y + c_4 y + c_5.
\end{aligned}$$

In this case, we obtain 7 Killing symmetries which are given by

$$X_0, \quad X_2, \quad X_3, \quad X_5, \quad (27)$$



$$X_{14} = \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z}, \quad (28)$$

$$X_{15} = \frac{1}{2} \left( e^{-2x} - y^2 + z^2 \right) \frac{\partial}{\partial y} + y \left( \frac{\partial}{\partial x} - z \frac{\partial}{\partial z} \right), \quad (29)$$

$$X_{16} = \frac{1}{2} \left( e^{-2x} + y^2 - z^2 \right) \frac{\partial}{\partial z} + z \left( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right). \quad (30)$$

### 3.5 Self Similar Solution of Infinite Kind for Parallel Dust Case

This case yields the solution of the Killing equations

$$\begin{aligned} k^0 &= \frac{1}{x} \left[ e^{-t} (c_1 y + c_3 z + c_5) - e^t (c_2 y + c_4 z + c_6) + c_{10} x \right], \\ k^1 &= e^{-t} (c_1 y + c_3 z + c_5) + e^t (c_2 y + c_4 z + c_6), \\ k^2 &= -e^{-t} c_1 x - e^t c_2 x + c_7 z + c_8, \\ k^3 &= -e^{-t} c_3 x - e^t c_4 x - c_7 y + c_9, \end{aligned}$$

The corresponding Killing symmetries are:

$$X_0, \quad X_2, \quad X_3, \quad X_5, \quad (31)$$

$$X_{17} = e^{-t} \left[ \frac{1}{x} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right], \quad (32)$$

$$X_{18} = -e^t \left[ \frac{1}{x} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right], \quad (33)$$

$$X_{19} = e^{-t} \left[ \frac{y}{x} \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right], \quad (34)$$

$$X_{20} = e^{-t} \left[ \frac{z}{x} \frac{\partial}{\partial t} + z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right], \quad (35)$$

$$X_{21} = -e^t \left[ \frac{y}{x} \frac{\partial}{\partial t} - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right], \quad (36)$$

$$X_{22} = -e^t \left[ \frac{z}{x} \frac{\partial}{\partial t} - z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \right]. \quad (37)$$

Now we investigate the corresponding Noether symmetries for all the cases mention above.

## 4 Noether Symmetries of Plane Symmetric Spacetime

The Lagrangian for the Plane symmetric spacetime (7) is

$$L = A(x)\dot{t}^2 - \dot{x}^2 - B(x)(\dot{y}^2 + \dot{z}^2). \quad (38)$$

Using Eq.(4) and after separation of monomials, the determining equations turn out to be

$$\xi_{,t} = 0, \quad \xi_{,x} = 0, \quad \xi_{,y} = 0, \quad \xi_{,z} = 0, \quad (39)$$

$$\xi_{,s} - 2\eta^1_{,x} = 0, \quad \eta^2_{,z} + \eta^3_{,y} = 0, \quad 2\eta^1_{,s} + G_{,x} = 0, \quad (40)$$

$$A[2\eta^0_{,t} - \xi_{,s}] + A_{,x}\eta^1 = 0, \quad (41)$$

$$B[2\eta^2_{,y} - \xi_{,s}] + B_{,x}\eta^1 = 0, \quad (42)$$

$$B[2\eta^3_{,z} - \xi_{,s}] + B_{,x}\eta^1 = 0, \quad (43)$$

$$A\eta^0_{,x} - \eta^1_{,t} = 0, \quad A\eta^0_{,y} - B\eta^2_{,t} = 0, \quad A\eta^0_{,z} - B\eta^3_{,t} = 0, \quad (44)$$

$$\eta^1_{,y} + B\eta^2_{,x} = 0, \quad \eta^1_{,z} + B\eta^3_{,x} = 0, \quad (45)$$

$$2B\eta^2_{,s} + G_{,y} = 0, \quad 2B\eta^3_{,s} + G_{,z} = 0, \quad (46)$$

$$2A\eta^0_{,s} - G_{,t} = 0, \quad G_{,s} = 0. \quad (47)$$

This is a system of 19 differential equations with six unknowns namely  $\xi$ ,  $\eta^0$ ,  $\eta^1$ ,  $\eta^2$ ,  $\eta^3$  and  $G$ . Now we investigate the Noether symmetries in the presence of gauge term by solving the above differential equations for the cases discussed in the previous section.

### 4.1 Minkowski Spacetime

For Minkowski spacetime the Lagrangian becomes

$$L = \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \quad (48)$$

Solving Eqs.(39)-(47) simultaneously by taking  $A(x) = B(x) = 1$ , we obtain

$$\begin{aligned} \xi &= \frac{1}{2}c_1s^2 + c_2s + c_3, \\ \eta^0 &= \frac{1}{2}[(c_1t + c_7)s + c_2t] + c_9x + c_{15}y + c_{16}z + c_{18}, \\ \eta^1 &= \frac{1}{2}[(c_1x - c_4)s + c_2x] + c_9t + c_{12}y + c_{10}z + c_{11}, \end{aligned}$$

$$\begin{aligned}
\eta^2 &= \frac{1}{2}[(c_1 y - c_6)s + c_2 y] + c_{15}t - c_{12}x + c_{13}z + c_{14}, \\
\eta^3 &= \frac{1}{2}[(c_1 z - c_8)s + c_2 z] + c_{16}t - c_{10}x - c_{13}y + c_{17}, \\
G &= \frac{1}{2}(t^2 - x^2 - y^2 - z^2)c_1 + c_7 t + c_4 x + c_6 y + c_8 z + c_5.
\end{aligned}$$

In this case we get 17 Noether symmetries. It is an interesting fact that 10 symmetries are exactly the same as the Killing symmetries while the remaining 7 are

$$X_{23} = \frac{\partial}{\partial s}, \quad X_{24} = s \frac{\partial}{\partial s} + \frac{1}{2} \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right), \quad (49)$$

$$X_{25} = \frac{1}{2}s \left( s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right), \quad X_{26} = \frac{1}{2}s \frac{\partial}{\partial t}, \quad (50)$$

$$X_{27} = -\frac{1}{2}s \frac{\partial}{\partial x}, \quad X_{28} = -\frac{1}{2}s \frac{\partial}{\partial y}, \quad X_{29} = -\frac{1}{2}s \frac{\partial}{\partial z}. \quad (51)$$

Thus we get additional symmetries of spacetime using Noether approach. Symmetry generator  $X_{23}$  indicates translation in  $s$  and it is always present in the case of the Lagrangian of minimizing the arc length [31] while  $X_{24}$  represents a scaling symmetry in  $s$ ,  $t$ ,  $x$ ,  $y$  and  $z$  direction. This symmetry is an important one as it may be used to eliminate the  $s$  dependence in the generators given in Eq.(50) and Eq.(51). It is mentioned here that Hussain et al. [18] investigated Noether symmetries of Minkowski spacetime but with spherical polar coordinates and obtained a 17 dimensional Lie algebra.

## 4.2 Taub's Universe

The Lagrangian for the Taub's spacetime takes the form

$$L = x^{-2/3} \dot{t}^2 - \dot{x}^2 - x^{4/3} [\dot{y}^2 + \dot{z}^2]. \quad (52)$$

The simultaneous solution of differential equations (39)-(47) takes the form

$$\begin{aligned}
\xi &= c_1 s + c_2, \quad \eta^0 = \frac{2}{3} c_1 t + c_6, \quad \eta^1 = \frac{1}{2} c_1 x, \\
\eta^2 &= \frac{1}{6} c_1 y + c_3 z + c_4, \quad \eta^3 = \frac{1}{6} c_1 z - c_3 y + c_5
\end{aligned}$$

and the gauge term turns out to be constant here which can be taken as zero without the loss of any generality. This solution forms a six dimensional

algebra. Here 4 symmetries are exactly the same as Killing symmetries while the 2 additional symmetries are

$$X_{23} = \frac{\partial}{\partial s}, \quad X_{30} = s \frac{\partial}{\partial s} + \frac{2}{3} t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x} + \frac{1}{6} y \frac{\partial}{\partial y} + \frac{1}{6} z \frac{\partial}{\partial z}. \quad (53)$$

As before, these two additional symmetries are translation in  $s$  and scaling in  $s$ ,  $t$ ,  $x$ ,  $y$  and  $z$ .

### 4.3 Anti-deSitter Universe

For anti-deSitter universe, the Lagrangian is given by

$$L = e^{2x} [\dot{t}^2 - \dot{y}^2 + \dot{z}^2] - \dot{x}^2. \quad (54)$$

In this case, the solution of determining equations becomes

$$\begin{aligned} \xi &= c_1, & \eta^1 &= c_2 t + c_3 z + c_5 y + c_4, \\ \eta^0 &= \frac{1}{2} (-e^{-2x} - t^2 - y^2 - z^2) c_2 - c_3 z t - c_5 y t - c_4 t + c_6 y + c_9 z + c_{11}, \\ \eta^2 &= \frac{1}{2} (e^{-2x} - t^2 - y^2 + z^2) c_5 - c_2 t y - c_3 z y - c_4 y + c_6 t + c_8 z + c_7, \\ \eta^3 &= \frac{1}{2} (e^{-2x} - t^2 + y^2 - z^2) c_3 - c_2 t z - c_5 y z - c_4 z + c_9 t - c_8 y + c_{10}. \end{aligned}$$

The guage term is also zero here. This solution forms 11 dimensional Lie algebra. In this case, the Killing algebra of anti-deSitter universe is also a subalgebra of Noether symmetries. We obtain only one additional symmetry here which is a translation in  $s$ , i.e.  $\partial/\partial s$ .

### 4.4 Self Similar Solution of Infinite Kind for Parallel Perfect Fluid Case

Here the Lagrangian takes the form

$$L = \dot{t}^2 - \dot{x}^2 - e^{2x} [\dot{y}^2 + \dot{z}^2]. \quad (55)$$

Thus the solution of the determining equations becomes

$$\xi = c_1, \quad \eta^0 = \frac{1}{2} c_2 s + c_{10},$$

$$\begin{aligned}
\eta^1 &= c_6 y + c_4 z + c_5, \quad G = c_2 t + c_3, \\
\eta^2 &= \frac{1}{2}(e^{-2x} - y^2 + z^2)c_6 - c_5 y - c_7 z - c_4 z y + c_9, \\
\eta^3 &= \frac{1}{2}(e^{-2x} + y^2 - z^2)c_4 - c_5 z - c_6 z y + c_7 y + c_8.
\end{aligned}$$

In this case the gauge function  $G$  is a linear function of time. This solution forms a 9 dimensional Lie algebra. Out of nine, seven symmetries are same as the killing symmetries while the additional symmetries are

$$X_{23} = \frac{\partial}{\partial s} \quad \text{and} \quad X_{31} = \frac{s}{2} \frac{\partial}{\partial t}.$$

#### 4.5 Self Similar Solution of Infinite Kind for Parallel Dust Case

The Lagrangian for self similar solution of infinite kind for parallel dust case is

$$L = x^2 \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \quad (56)$$

This case yields the solution of the determining equations

$$\begin{aligned}
\xi &= \frac{1}{2}c_1 s^2 + c_2 s + c_3, \\
\eta^0 &= \frac{1}{x}[e^{-t}(-c_8 \frac{s}{2} + c_{10}y + c_{11}z + c_{13}) - e^t(c_9 y + c_{12}z - c_7 \frac{s}{2} + c_{14}) + c_{18}x], \\
\eta^1 &= e^{-t}(-c_8 \frac{s}{2} + c_{10}y + c_{11}z + c_{13}) + e^t(-c_7 \frac{s}{2} + c_9 y + c_{12}z + c_{14}) + \frac{1}{2}x(c_1 s + c_2), \\
\eta^2 &= \frac{1}{2}s(c_1 y - c_4) - c_{10}x e^{-t} - c_9 x e^t + \frac{1}{2}y c_2 + c_{15}z + c_{16}, \\
\eta^3 &= \frac{1}{2}s(c_1 z - c_6) - c_{11}x e^{-t} - c_{12}x e^t + \frac{1}{2}z c_2 - c_{15}y + c_{17}
\end{aligned}$$

and the gauge term turns out to be

$$G = -\frac{1}{2}(x^2 + y^2 + z^2)c_1 + c_4 y + c_6 z + c_5 + c_7 x e^t + c_8 x e^{-t}.$$

This solution also yields a 17 dimensional Lie algebra in which 10 symmetries are same as Killing symmetries while the remaining 7 are

$$X_{23} = \frac{\partial}{\partial s}, \quad X_{32} = s \frac{\partial}{\partial s} + \frac{1}{2}(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}), \quad (57)$$

$$X_{33} = \frac{1}{2}s(s\frac{\partial}{\partial s} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}), \quad (58)$$

$$X_{28} = -\frac{s}{2}\frac{\partial}{\partial y}, \quad X_{29} = -\frac{s}{2}\frac{\partial}{\partial z}, \quad (59)$$

$$X_{34} = -\frac{se^{-t}}{2}(\frac{1}{x}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}), \quad X_{35} = \frac{se^t}{2}(\frac{1}{x}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}). \quad (60)$$

## 5 Summary and Conclusion

The main purpose of this paper is to investigate Killing and Noether symmetries of static plane symmetric spacetime. For this purpose we have considered five different cases namely Minkowski spacetime, Taub's universe, anti-deSitter universe and self similar solutions of infinite kind for parallel perfect fluid and dust cases. The Killing and Noether symmetries for each case are given in the following table.

**Table 1. Comparison of Killing and Noether Symmetries**

Metric	Killing	Noether
Minkowski Spacetime	10	17
Taub's Universe	4	6
Anti-deSitter Universe	10	11
Self Similar Solution (Perfect Fluid Case)	7	9
Self Similar Solution (Dust Case)	10	17

It is mentioned here that in all the cases, Killing symmetries are contained in Noether symmetries. Thus we obtain additional symmetries of the above mentioned spacetimes by finding the Noether generators. Moreover, the scalar curvature is constant (zero and non-zero) in all cases. Thus we conclude that the Killing symmetries of static plane symmetric spacetime always form a sub algebra of the Noether symmetries. These examples not only provides Killing and Noether symmetries but also validate the conjecture given by [1].

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